Combinatorics Tips: Part II

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1 Local to Global Style Arguments

Really cleanly write out your local observations, then find a good way to tightly use them globally!
Invariants and monovariants!

- Invariants and monovariants: At each step, what remains the same or is guaranteed to increase (e.g. mod n, colourings, orders, spatial properties, counts, etc.)?
- Greedy/extremal proofs of existence: What natural quantity can you always increase or decrease at every "bad" instance?
- Disallowed sub-configurations: What are they and how can we infer some global bound from them without much of a loss?

Mindsets to get into: invariants, monovariants, greedy/extremal

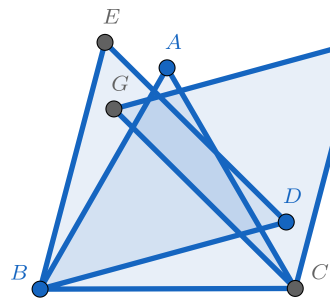
Example 1. *(ISL 2019 C7) Consider* 2018 *pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular* edges *that meet at* vertices*. Notice that there are an even number of vertices on each circle. Given a circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice – once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least* 2061 *yellow points, then the vertices of some region are all yellow.*

Example 2. *(ISL 2007 C8) Let P be a convex polygon with n vertices. Prove that there are at most $\frac{2n}{3}$ equilateral triangles with side length 1 formed by vertices of P.*

The key constraint in this problem is that the points of P are convex. The problem is essentially asserting that the fact that the triangles are unit equilateral triangles should contradict convexity as soon as the triangles are too numerous. A natural first step is to see if any local configurations of unit equilateral triangles violate convexity. Playing around with small cases of unit equilateral triangles sharing vertices, we can arrive at the following impossible configuration:

- ABC is a unit equilateral triangle of P; and
- there are two unit equilateral triangles of P passing through B and C, respectively, and entirely contained in the halfplane of BC containing A.

This is impossible since the two triangles through B and C each contain a point above the line parallel to BC through A and that the angle formed by these vertices and A is concave.



This kind of observation is exactly the kind of local observation that is highly restrictive and should say something global about the configuration. Optimistically, this observation might be enough to capture everything this problem needs about its primary complicated objects: (1) the convexity of P; and (2) the fact that the triangles are unit equilateral triangles. The question now is how to deduce a global bound on the number of triangles from this disallowed configuration alone? Is this even possible without using more about (1) and (2)?

This is the point when it would be useful to try to construct examples to see whether or not this is possible. Attempting this for a bit should convince you that we probably can deduce an upper bound on the number of triangles. What is a good candidate for the segment BC to try to force the disallowed configuration? Observe that GC and BD are both "above" BC. Thus a good candidate BC is the lowest edge in the configuration. Here, the notion of lowest needs to be made more formal: consider the edge BC such that its midpoint has lowest y coordinate. A little casework shows that this definition implies one of B or C cannot be part of a second unit equilateral triangle. Let's now turn this observation into a concrete bound.

We will now show that this gives a suboptimal upper bound of n. We give two arguments based on viewing vertices part of a unit equilateral triangle as connected in a graph and another clean recursive/inductive approach.

First Argument for $\leq n$. For each unit equilateral triangle, keep its lowest edge i.e. the edge with midpoint of minimal y coordinate. These edges induce a graph G on the vertices of P. If G contains a cycle, then the lowest edge BC of this cycle yields the configuration above. Thus there are at most $n-1$ edges in G. Since this is also the number of unit equilateral triangles, we are done. □

Second Argument for $\leq n$. Take the lowest edge BC among all unit equilateral triangles. Either B or C must be in exactly one equilateral triangle. Remove this vertex and the corresponding triangle. The claim now follows by induction. □

The second argument is essentially assigning each triangle to a different vertex. Can these arguments be tweaked to produce our desired bound of $2n/3$? The answer unfortunately seems to be no – playing around with variants of these arguments doesn't seem to produce $2n/3$. To get a bound that even looks like $2n/3$, we would need to do something like assign each vertex at most twice and have each equilateral triangle be assigned to at least three times. However, it is not clear how to make this work with anything resembling our current argument.

Why might we expect something like this? Whenever we fix a direction, we are not assigning any vertices that never are part of a lowest edge of a triangle. For many configurations, this is a huge number of vertices.

Furthermore, our extremal argument is mismatched to our disallowed configuration. In order to specify the smallest edge, we needed to fix a direction for the y axis and hack together a definition of "lowest". The disallowed configuration

For many hard problems, it is possible to be off by a subtle point but have this be the hard part

Local to global might not be efficient enough

Example 1. *In a group of $2n+1$ people, each pair is classied as friends or strangers. For every set S of at most n people, there is one person outside of S who is friends with everyone in S. Prove that at least one person is friends with everyone else.*

The most obvious application of an extremal argument is to consider the vertex of highest degree. However, this approach quickly gets messy and does not seem to be easy to relate to the condition in the problem. The condition seems to guarantee a large number of edges, so it makes sense to try to globally count the number of edges and apply a form of pigeonhole. However, the average number of edges guaranteed is not as high as $2n$ so this approach cannot work out. Based on this, it makes more sense to try to work locally in the graph to guarantee a large number of edges. A natural idea is to show that there is a large clique in the graph. This is concretely relates to the condition in the problem, which guarantees that cliques of size at most n can be increased in size. Now, considering the maximal clique reveals exactly enough structure to solve the problem.

Solution: Consider the largest clique C in the graph with vertices V. If C contains at most n vertices, then there must be another vertex adjacent to all of the vertices in C, which contradicts its maximality. Therefore $|C| \geq n+1$ and there is a vertex $v \in C$ that is adjacent to all vertices in $V \backslash C$ since at most n vertices are not in C. Therefore v is adjacent to all other vertices in the graph. □

Say to do this as an exercise (if don't have time to write up)

Example 3. *(IMO 2011) Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.*

Example 4. *(ISL 2016) There are $n \geq 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.*

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

2 Restrict Your Choices: Structuring Direct/Greedy Approaches

Also conditions or extraneous degrees of freedom are a major theme on the IMO!

Example 5. *(Russia 2005) In each of 100 boxes, there are some number of apples and oranges. Prove that you can choose 34 boxes in such a way that at least one third of all apples and at least one third of all oranges are in the chosen boxes.*

Give both greedy and alternating sum solutions

Example 6. *(Russia 2006) A* 3000×3000 *square board has been cut into dominoes. Show that the dominoes can be painted in three colors such that the number of dominoes of each color is the same and each domino has at most two neighboring dominoes of the same color.*

It is difficult to work with domino tilings directly in general since we do not really know too much about how they are distributed in the tiling but at the same time the fact that it is a domino tiling is an important constraint. A natural idea is to try to apply an algorithmic approach, but this quickly becomes complicated as cyclic dependencies make it difficult to develop a simply-stated algorithm. It also becomes difficult to ensure that there are the same number of dominoes of each color and difficult to make sure that the algorithm actually uses something substantial about domino tilings. Instead, since the condition that each domino has at most two neighboring dominoes of the same color is a relatively weak condition, we try to impose a structure to our coloring that depends mostly on the underlying square board and not on the particular tiling to simplify the problem (this is "wishful thinking"). One reasonable way to do this, while still using the fact that the tiling is a domino tiling, is to color the black squares and use the local observation that each domino passes through exactly one black square. We make use of this in the following solution:

Solution: Consider a chessboard coloring of the square board and consider all of the black squares in the coloring ($i + j$ is odd). Now color each black square with a number in $\{0, 1, 2\}$ such that if (i, j) are the coordinates of a black square then it is colored with the number $i - j \pmod 3$. Now color each domino with the color of the unique black square it contains. This ensure that there are the same number of each color, since the dimensions of the board are divisible by 6. Given a domino A, the dominoes that can neighbor A must pass through one of the 8 closest black squares to the black square passing through A. Among these 8 black squares, exactly two have the same color as A, implying the result. □

Example 6. *(IMO 2014) For each positive integer n, the Bank of Cape Town issues coins of denomination* $\frac{1}{n}$*. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most* $99 + \frac{1}{2}$*, prove that it is possible to split this collection into* 100 *or fewer groups, such that each group has total value at most* 1*.*

Example 7. *(IMO 2017)*

3 Problems

Only some of these problems are related to the ideas discussed. The main point is just to get some combinatorics practice in before the IMO! Many of these are very hard so feel free to ask for hints. They also may not be in increasing order of difficulty.

A1. Initially, there are 111 pieces of clay on the table of equal mass. In one turn, you can choose several groups of an equal number of pieces and push the pieces into one big piece for each group. What is the least number of turns after which you can end up with 11 pieces no two of which have the same mass?

A2. There are stones with a total mass of 9 tons that should be transported by trucks. None of the stones is heavier than 1 ton and each vehicle has a capacity of 3 tons. Determine the minimum number of necessary trucks such that the stones can be transported at the same time for sure.

A3. At a math party, every person knew exactly three other people. Prove that the people at the party can be put into two rooms such that each person knew at most one other person in his or her room.

A4. Meanwhile at a different math party, some pairs of people shook hands. Two people are close if there is another person with whom they both shook hands. Prove that there are two people who are close and shook hands with the same number of people.

A5. There are 100 distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

A6. There are three colleges in a town. Each college has n students. Any student of any college knows a total of $n+1$ students from the other two colleges. Prove that it is possible to choose a student from each of the three colleges so that all three students know each other.

A7. Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \le k \le 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

B1. A circle has been cut into 2000 sectors. There are 2001 frogs inside these sectors. There will always be some two frogs in the same sector; two such frogs jump to the two sectors adjacent to their original sector (in opposite directions). Prove that, at some point, at least 1001 sectors will be inhabited.

B2. In a city's bus route system, routes are sets of stops. Furthermore, any two routes share exactly one stop and every route includes at least four stops. Prove that the stops can be partitioned into two groups such that each route contains stops from each group.

B3. A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(a) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(b) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

B4. Let $S = \{x_1, x_2, \ldots, x_{k+l}\}$ be a $(k+l)$-element set of real numbers contained in the interval $[0,1]$; k and l are positive integers. A k-element subset $A \subset S$ is called nice if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l}\dbinom{k+l}{k}$.

B5. There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n-1$ is not divisible by 3.

B6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.

B7. On the cartesian plane are drawn several rectangles with the sides parallel to the coordinate axes. Assume that any two rectangles can be cut by a vertical or a horizontal line. Show that it's possible to draw one horizontal and one vertical line such that each rectangle is cut by at least one of these two lines.

C1. In a country with n cities, some pairs of cities are connected by one-way flights operated by one of two companies A and B. Two cities can be connected by more than one flight in either direction. An AB-word w is called implementable if there is a sequence of connected flights whose companies' names form the word w. Given that every AB-word of length 2^n is implementable, prove that every finite AB-word is implementable.

C2. Let G be a connected graph such that every vertex has degree at least 3. Prove that there is a cycle such that removing the edges of this cycle does not disconnect the graph.

C3. Let n be a positive integer, and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word, consisting of just letters a and/or b. Suppose that the minimal period N of W is greater than 2^n. A finite nonempty word U is said to appear in W if there exist indices $k \le \ell$ such that $U = x_k x_{k+1} \ldots x_\ell$. A finite word U is called ubiquitous if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

C4. Let $a_1, a_2, \ldots, a_n$ be a sequence of distinct positive real numbers. For the next n days, there will be a snowstorm each day covering all positions on the x-axis with value ≥ 0 or all positions with value ≤ 0. Sarah starts at position 0 and knows where each snowstorm will be. Before each snowstorm, Sarah will jump some distance to one side to make sure she avoids it (even if she was already avoiding it). Prove that no matter where the snowstorms are going to be, it is always possible for Sarah to do this using exactly one jump of length i for each $i \in \{1, 2, \ldots, n\}$.

C5. There are some markets in a city. Some of them are joined by one-way streets, such that for any market there are exactly two streets to leave it. Prove that the city may be partitioned into 1014 districts such that streets join only markets from different districts, and by the same one-way for any two districts (either only from first to second, or vice-versa).

C6. Fix positive integers n and $k \geq 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after finitely many steps, you can reach a state where at least $n - k + 2$ of the numbers on the blackboard are simultaneously divisible by k.

C7. Let S be a subset of the plane containing at least three points and satisfying that if $A, B, C \in S$ are distinct then the circumcenter of ABC is in S. Prove that S contains a point within distance 1 of the origin.